LINEAR ALGEBRA

① Introduction
② Determinant
③ Rank of Matrix
④ System of Equations
⑤ Eigen values & Eigen vectors
⑥ Cayley Hamilton theorm.

(m×n) (row×column) a_{ij} i^{th} row & j^{th} column element.

Matrix

→ A arrangement real or complex no.s in rows or columns is known as matrix.

$m=n=$ square matrix
$m\neq n=$ rectangular matrix — $m>n$ vertical matrix; $m<n$ Horizontal matrix
$m=1$ — row matrix
$n=1$ = colum matrix

→ ($a_{ij}=0$ $i>j$) Upper triangular
→ ($a_{ij}=0$ $i<j$) Lower triangular
both → diagonal matrix $\begin{cases} a_{ij}=k_i \ \forall i=j \\ a_{ij}=0 \\ j\neq i \end{cases}$

→ $a_{ij}=k$ $i=j$; $a_{ij}=0$ $i\neq j$ } Scalar matrix
if $k=1$ ⇒ unit Matrix

→ $a_{ij}=0$ } Null matrix.

→ Idempotent Matrix ($A^2=A$) if square

→ Nilpotent Matrix ($A^m=0$) m = index of matrix } square. $\boxed{m\leq \text{order}}$
(both $|A|=0$, trace $A=0$)
eigen value $=0$.

→ Involutory Matrix $A^2=I$

→ Periodic $A^{m+1}=A$ $\boxed{\text{period } 1 \Rightarrow \text{Idempotent}}$
m → period of m.

- **Symmetric Matrix**

if $A^T = A$ ie) $A_{ij} = A_{ji}$ $\forall$ i,j < order   square !

- **Skew Symmetric Matrix**

$A^T = -A$ = $a_{ij} = -a_{ji}$ $\forall$ i, j < order   square !

diagonal elements compulsary zero.

- **Orthogonal Matrix**

$A \cdot A^T = I$ or $A^T = A^{-1}$

eg:- $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Properties

The diagonal elements of skew ~~ara~~ symmetric matrix is zero.

→ The diagonal elements of skew symmetric matrix is zero.

→ The sum of elements of skew symmetric matrix is defined as $a_{ij} = i^m - j^m$ $\forall i, j$

→ If a square matrix $A_{n \times n}$ is defined as

then A is skew symmetric.

$$\boxed{\begin{aligned} a_{ij} &= i^m - j^m \\ a_{ij} &= -(j^m - i^m) \\ a_{ij} &= -(a_{ji}) \end{aligned}}$$

→ The determinant of odd order skew symmetric is zero

$A^T = -A$

$|A^T| = -|A|$

$|A| = -|A|$

$\Rightarrow |A| = 0$

→ Determinant of ~~equal~~ even order skew symmetric is a perfect square.

→ Determinant of an orthogonal matrix is ± 1

$A^T = A^{-1}$

$|A| = \frac{1}{|A|}$

$|A|^2 = 1 \Rightarrow |A| = \pm 1$

→ The no. of independent elements in a symmetric matrix of $A_{n \times n}$ is $\frac{n(n+1)}{2}$

→ No. of independent elements in a skew symmetric matrix.

$$\frac{n(n+1)}{2} - n = \frac{n^2 + n - 2n}{2} = \frac{n(n-1)}{2}$$

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Scalar — 1 } independent
diagonal — n } element

→ Every Square matrix can be expressed as the sum of symmetric & skew skew symmetric matrix uniquely.

$$A = \left(\frac{A+A^T}{2}\right) + \left(\frac{A-A^T}{2}\right) \Big\} \text{ uniquely.}$$

$\downarrow$ symmetric, $\downarrow$ skew-symmetric.

A = Symmetric

Q) if $A_{5\times5} = [a_{ij}]$ such that $a_{ij} = i^2 - j^2 \; \forall \, i, j$

then $\left[\sum_{i,j=1}^{5} (a_{ij} = ?)\right]$ zero

Q Same quest $|A| = ?$ zero.

Q) if $A = \begin{bmatrix} 1 & 8 & -3 \\ 2 & 5 & 6 \\ 5 & -4 & 6 \end{bmatrix}$ then corresponding skew symmetric matrix is.

$$\left(\frac{A-A^T}{2}\right) = \begin{bmatrix} 0 & 3 & -4 \\ -3 & 0 & 5 \\ 4 & -5 & 0 \end{bmatrix}$$

Q) How many no. of symmetric matrices can be formed with $(0, \pm1, \pm2)$

n-order $\frac{n(n+1)}{2}$ – independent elements.

5-choices ∴ $5^{\left(\frac{n(n+1)}{2}\right)}$

→ each independent elements can take 5 possible values ∴ $\frac{n(n+1)}{2}$ elements $= 5^{\frac{n(n+1)}{2}}$

→ n element can take 5^n possible combination

⇓

Skew symmetric $5^{\frac{n(n-1)}{2}}$

diagonal 5^n

Scalar — 5

Q) How many symmetric. if {0, 1, 2}

$= 3^{\frac{n(n+1)}{2}}$

How many skew symmetric.

= 1 {all element zero}

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$\{0, \pm 1, 2\}$	$\{\pm 1, \pm 2\}$
Square $= 4^{\frac{n(n+1)}{2}}$	Square $= 4^{\frac{n(n+1)}{2}}$
Skew $= 3^{\frac{n(n-1)}{2}}$	Skew $= 0$

Q) $A_{3\times 2}$ $B_{2\times 4}$ AB how many multiplications ——?

Each element in result $= (AB)_{3\times 4}$ $\therefore$ $\underline{12 \times 2 = 24}$

is got by 2 multiplication and one addition.

$$\boxed{A_{m\times n} \ \& \ P_{n\times p} \text{ then } (A\cdot P)_{m\times p} \Rightarrow \begin{array}{l} mnp \text{ multiplication} \\ m(n-1)p \text{ additive} \end{array}}$$

Minor of an element. $[M_{ij}]$ Standard notation (only for square)

→ The determinant of the remaining matrix after removing i^{th} row & j^{th} column is called minor (M_{ij}) of the element a_{ij} of a square matrix.

Cofactor of a_{ij} :-

$$\text{Cofactor}(a_{ij}) = (-1)^{i+j} \times M_{ij}$$

Cofactor Matrix:

Every element replaced with it's cofactor.

$$\begin{bmatrix} M_{11} & -M_{12} & M_{13} \\ -M_{21} & M_{22} & -M_{23} \\ M_{31} & -M_{32} & M_{33} \end{bmatrix}$$

Adjoint of Matrix Adj (A)

$$\text{Adj } A = (\text{Cofactor of } A)^T$$

$$\begin{bmatrix} M_{11} & -M_{21} & M_{31} \\ -M_{12} & M_{22} & -M_{32} \\ M_{13} & -M_{23} & M_{33} \end{bmatrix}$$

Determinant of (nxn) matrix

function : (nxn ⟶ R)

eg:- $|A| = a_{11}M_{11} + a_{12}(-M_{12}) + a_{13}(M_{13})$

Sum of elements of a $\boxed{\text{row}}$ multiplied with corresponding cofactors

OR Sum of elements of a $\boxed{\text{column}}$ multiplied by corresponding cofactors.

2n Way to expand determinant. n! terms in a determinant. each term has n multiplication factors ∴ n.n! total multiplication required. n!-~~multiplication~~ addition. required.	$A_{n\times n}$

Inverse of a matrix

if $AB = BA = I$ then A is inverse of B
then B is inverse of A
then A & B are called inverse matrices to each other.

$\Rightarrow A^{-1} = B$

$\Rightarrow B^{-1} = A$

Property

$\boxed{\text{Multiplication associative}}$

- $(AB)^{-1} = B^{-1}A^{-1}$
- $(ABCD)^{-1} = \cancel{(DCBA)^{-1}}\ \underline{D^{-1}C^{-1}B^{-1}A^{-1}}$

$((AB)\cdot(CD))^{-1}$
$= (CD)^{-1}\cdot(AB)^{-1}$
$= \underline{D^{-1}C^{-1}B^{-1}A^{-1}}$

- $A\dfrac{(\text{Adj}A)}{|A|} = I = \dfrac{\text{Adj}(A)}{|A|}A$

$\Rightarrow \boxed{A^{-1} = \dfrac{\text{Adj}(A)}{|A|}} \Rightarrow \boxed{[\text{Adj}\,A]^{-1} = \dfrac{A}{|A|}}$

if $|A| = 0 \Rightarrow$ A is singular matrix $\underline{A^{-1}\text{ doesn't exist}}$.
A is not invertable

- If $|A| \neq 0 \Rightarrow$ A is non singular matrix

Q) $A = \begin{bmatrix} 1 & -3 & 2 \\ 0 & -1 & 5 \\ 1 & 2 & 4 \end{bmatrix}$ $\boxed{A^{-1} = ?}$ $|A| = 1(-4-10) + 3(-5) + 2(+1)$

$|A| = -14 - 15 + 2 = \underline{-27}$.

Cofactor $\begin{bmatrix} -14 & 5 & 1 \\ 16 & 2 & -5 \\ 13 & -5 & -1 \end{bmatrix}$ Adj $= \begin{bmatrix} -14 & 16 & 13 \\ 5 & 2 & -5 \\ 1 & -5 & -1 \end{bmatrix}$

Q) $A = \begin{vmatrix} 1 & -1 & 3 \\ -2 & 0 & 4 \\ 2 & -2 & 6 \end{vmatrix}$ Adj A =

$|A| = 0$

Since $R_3 = 2R_1$

$Co\,A = \begin{vmatrix} +8 & 4 & 4 \\ 0 & 0 & 0 \end{vmatrix}$

Properties

- Determinant of diagonal, triangular, & scalar is product of diagonal elements.
- If any two rows or columns are identical then its determinant is zero.
- If any two rows or columns are proportional then determinant is zero.
- If any two $\frac{\text{rows}}{\text{columns}}$ are interchanged then determinant is absolutely same with different sign.
- Adding a row to ~~any~~ another row ~~with~~ after multiplied by a non zero scalar. then determinant remains same.

$R_j \rightarrow R_j + kR_i \Rightarrow |A| = \text{same}$.

if any $\frac{\text{row}}{\text{column}}$ is multiplied by k then determinant becomes k times

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- $|I_{n\times n}| = 1$
- $|AB| = |A|\cdot|B|$
- $|KA| = K^n |A| \quad (A_{n\times n})$
- $|A^n| = |A|^n$
- $|A| = |A^T|$
- $|A^{-1}| = \frac{1}{|A|}$

- $|\text{adjoint of }(A)| = |A|^{n-1}$

$$\boxed{\begin{aligned} &\text{adj}(A) = |A|\, A^{-1} \\ &\Rightarrow |\text{adj}(A)| = |A|^n\, |A^{-1}| \\ &\Rightarrow |\text{adj}(A)| = |A|^n \cdot \frac{1}{|A|} \\ &\Rightarrow |\text{adj}(A)| = |A|^{n-1} \end{aligned}}$$

$$(\text{Adj } A)^{-1} = \frac{\text{Adj}(\text{Adj } A)}{|\text{Adj } A|}$$

- $|\text{Adj}(A^{-1})| = |(\text{Adj}(A))^{-1}| = |A|^{1-n}$
- $|\text{adj}(KA)| = K^{n(n-1)} |A|^{n-1}$
- $|\text{adj}(\text{adj } A)| = |A|^{(n-1)^2}$

Q) $\begin{vmatrix} x & a & a & a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix} = ?$

$C_1 \to C_1 + C_2 + C_3 + C_4$

$$\begin{vmatrix} x+3a & a & a & a \\ x+3a & x & a & a \\ x+3a & a & x & a \\ x+3a & a & a & x \end{vmatrix}$$

$R_2 - R_1$; $R_3 - R_1$; $R_4 - R_1$

$$\Rightarrow \begin{vmatrix} x+3a & a & a & a \\ 0 & (x-a) & 0 & 0 \\ 0 & 0 & (x-a) & 0 \\ 0 & 0 & 0 & (x-a) \end{vmatrix}$$

try to reduce any one row or column to a single element row/column with rest all zeroes.

$|A| = (x+3a)(x-a)^3$

• A matrix is obtained from $A(m \times n)$ by leaving some rows or columns or both. SUB MATRIX

No. of possible sub matrix = ~~(m+n) + (m+n)(m+n-1)/2~~ $1 + {}^{(m+n)}C_1 + {}^{m+n}C_2 + \dots + {}^{m+n}C_{m+n}$
$= 2^{m+n}$ ★★★

Every matrix is sub matrix of itself.

Minor of matrix

The determinant of a square sub matrix is called minor of the matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

$\begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \end{vmatrix}$ 3x3 minor.

$\begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix}$ 2x2 minor.

$|a_{12}|$ 1x1 minor.

Note:- Minor of a matrix is not unique but minor of an element is unique. (only sqr matrix)

no. of 3x3 minor in a matrix of $A_{pq} = {}^pC_3 \times {}^qC_3$

In general no. of r x r minors in $[A_{m\times n}]$ matrix $= {}^mC_r \cdot {}^nC_r$

→ for r x r minor r rows must be selected from m total rows ($\to {}^mC_r$ ways).
r columns must be selected from n total columns. ($\to {}^nC_r$)

eg:- $(1\times1) = {}^3C_1 \times {}^4C_1 = m\cdot n = \underline{12}$
$(2\times2) = {}^3C_2 \times {}^4C_2 = 3\times6 = 18$
$(3\times3) = {}^3C_3 \times {}^4C_3 = 1\times4 = \underline{4}$
$\underline{\underline{34}}$

Rank of Matrix $\rho(A)$ or $r(A)$

all minor $= 0$ then Vanishing.

- An order of highest non vanishing minor (non zero) of a matrix $A_{m\times n}$ is called rank of matrix
- An non negative integer r is said to be rank of matrix if atleast one $r\times r$ minor is non zero. (Given $r+1$ order minor doesnot exist) if it exist then it should be zero.
- For square matrix if determinant is non zero then rank $=$ order $\boxed{(n)}$ (because order $= n$ $n\times n$ minor is vanishing otherwise)

eg:- $\begin{bmatrix} 3 & 4 & -5 \\ 0 & 1 & 3 \\ -1 & 2 & 4 \end{bmatrix}$

$3\times(4-6)-4(\cancel{+0}\,3)-5(+1)$

$= -6 -12-5 = -23 \neq 0$

rank $= 3$.

eg. $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 4 & 3 \\ 5 & 8 & -5 \end{bmatrix}$

$C_3 = -C_1$

$|A| = 0$

rank < 3

$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4-6 = -2 \neq 0$

$\therefore$ rank $= 2$.

eg. $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

rank $= 2$

Elimentary transformations

1. Interchanging of any two rows. $R_i \leftrightarrow R_j$, $C_i \leftrightarrow C_j$ $\Rightarrow$ sign changes of det.

2. Multiplication of a row or column with a non zero scalar.
   $R_i \longrightarrow KR_i$ Det $\Rightarrow$ K times

3. Adding of a row to another row after multiplication by a non zero scalar $R_j \longrightarrow R_j + KR_i$ (det not changing)

Equalent Matrices ($\sim$)

→ Two matrices A and B are equalent to each other if one is obtained from other, by applying a sequence of elementary transformations.

→ $\boxed{\text{Rank of equalent matrices are same.}}$

Elimentary Matrix

A matrix is obtained from unit matrix by applying (any) a single elementary transformation is called elimentary matrix.

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

elimentary matrix.

Echelon form

A matrix is said to be in echelon form if it satisfies

→ ① If zero row exist then it should be below of all non zero rows

→ ② The no. of zeroes before first non zero element in every row ~~in every row~~ is less than such no of zeroes in the next row.

eg:-

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Always diagonal matrix & upper triangular matrix is in echlon form.

Note: Echelon form of a square matrix is an upper triangular matrix

- Every matrix can be reduced to echelon form by applying elementary row transformations.

- If an matrix is in echelon form then rank of a matrix is equal to the no. of non zero rows. [or equal to no. of independent rows/independent columns]

$$\boxed{\text{No. of independent row/column} = \text{Rank of } A}$$

Properties

→ Rank of matrix is non negative integer (whole number)

→ $\rho(0 \text{ matrix}) = 0$

→ rank of unit matrix – I $\rho(I) = n$ $\rho(I_{n\times n}) = n$

→ $\rho(A_{m\times n}) \leq \min(m, n)$ $\boxed{\rho(A) = \rho(A^T)}$

→ $\rho(A_{n\times n}) = n$ if $|A_{n\times n}| \neq 0$ Non singular

→ $\rho(A_{n\times n}) < n$ if $|A_{n\times n}| = 0$ Singular.

→ $\rho(A+B) \leq \rho(A) + \rho(B)$

→ $\rho(A-B) \geq \rho(A) - \rho(B)$

→ $\rho(AB) \leq \text{Min of } (\rho(A), \rho(B))$

→ $\rho(A_{n\times n}) = n \Rightarrow \rho(\text{Adj } A) = n$

→ $\rho(A_{n\times n}) = n-1 \Rightarrow \rho(\text{Adj } A) = 1$

→ $\rho(A_{n\times n}) \leq n-1 \Rightarrow \rho(\text{Adj } A) = 0$

eg:- $\rho(A_{6\times 6}) = 4 \Rightarrow \rho(\text{Adj } A) = 0$

$\rho(A_{5\times 5}) = 4 \Rightarrow \rho(\text{Adj } A) = 1$

Q) $A \times \begin{bmatrix} 3 & 4 & -5 \\ 2 & -1 & 3 \\ 7 & 6 & -5 \end{bmatrix} = \begin{bmatrix} 3 & 4 & -5 \\ 2 & -1 & 3 \\ 4 & 2 & 0 \end{bmatrix}$, then $A = ?$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

RHS = $R_3 \rightarrow R_3 - R_1$

$\therefore \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $R_3 \rightarrow R_3 - R_1$

$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

elimentary matrix obtained by applying same transformation

every elimentary row transformation is nothing but matrix multiplied with corresponding elementary matrix

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Q) $\begin{bmatrix} 1 & 7 & 5 \\ 2 & 4 & 3 \\ -1 & 0 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 7 & 3 \\ 2 & 4 & -1 \\ -1 & 0 & 4 \end{bmatrix}$

$AB = C$

$C \Rightarrow A(C_3 \rightarrow C_3 - 2C_1)$

$\therefore \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $C_3 \rightarrow C_3 - 2C_1$

$\Rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = B$

every elimentary column transformation is nothing but the corresponding elementary matrix multiplied by the given matrix.

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Practice Qu

1) $A = \begin{bmatrix} 3 & p & p \\ p & 3 & p \\ p & p & 3 \end{bmatrix}$ rank $=1$ then $p = ?$

3 so that 3x3 vanish
2x2 vanish.

$\begin{bmatrix} 3 & p & p \\ p-3 & 3-p & 0 \\ p-3 & 0 & 3-p \end{bmatrix}$ two row zero $\Rightarrow$ $p-3=0$, $3-p=0$, $\underline{p=3}$.

② $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -3 & 3 \\ 1 & 3 & -1 \end{bmatrix}$ & $B = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ then $|AB - BA| =$

$|AB - BA| = |AB| - |BA| = |A||B| - |B||A|$

$|A| = (3-9)2 + (1 \cdot 3) = -12 + 3 = \underline{-9}$

$|B| = 1(-1) = -1$

$(AB-BA)^T = AB^T - BA^T$
$= B^TA^T - A^TB^T$
$= BA - AB$

$(AB-BA)^T = -(AB-BA)$

$\Rightarrow (AB-BA) =$ skew symmetric.

$\boxed{\text{A \& B are symmetric matrices then AB+BA is symmetric, AB}-\text{BA is skew symmetric}}$

$\therefore$ odd order skew symmetric matrix det $=0$ Hence $|AB-BA| = 0$

③ if $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ & $B = A^{-1}$ then the element in the second row third column of B.

$\Rightarrow$ 3rd row second colown of $\frac{\text{Cofactor}}{|A|} = \frac{-1}{|A|} = \frac{-1}{2}$

option c

$|A| = 1(1) + 1(1) = 2$

Linear Independent Vectors

→ The set of Vectors $\{x_1, x_2, x_3 \dots x_n\}$ is said to be linear independent vector set if there is no zero scalar such that its linear combination of vectors is zero.

ie) $k_1x_1 + k_2x_2 + k_3x_3 + \dots\dots + k_nx_n = 0$ linear combination.

then $\boxed{k_1 = k_2 = k_3 = \dots k_n = 0}$ linear independent.

if any $k \neq 0$ then dependent.
any

→ Linear dependent vector

The set $x_1, x_2, x_3, \dots\dots x_n$ are linearly dependent vectors if there exist a non zero scalar $k_i \neq 0$ such that

$k_1x_1 + k_2x_2 + k_3x_3 + \dots k_nx_n = 0.$

One vector can be expressed as linear combination of other vectors

Note.

→ if A is a square matrix of order n and $|A| = 0$ then all the rows and columns are linearly dependent.

→ if $|A| \neq 0$. Then all the rows and columns are linearly independent.

Note:
- Subset of Independent set of Vectors are definitely independent
- Subset of dependent set of Vectors may be dependent or independent.
- Super set of dependent set of Vectors is also dependent.
- Super set of independent set of Vectors may or may not be independent.

10) $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$ $a, b, c \neq 0$ (real no.)

then Rank of $(A) = \underline{?}$ $|A| = 0$ (Skew symmetric) with order $= 3$ (odd)

2×2 Minor of $A = \begin{vmatrix} -a & 0 \\ -b & -c \end{vmatrix} = ac \neq 0$

then $\underline{\text{rank} = 2}$. $\underline{\text{option c}}$.

if condition $a, b, c \neq 0$ is not specified
possible rank $= 0$ (for $a, b, c = 0$ all zero)
rank $= 2$ (for atleast one non zero)

→ For symmetric & skew symmetry ~~the RANK is~~ of 3rd order RANK of 1 is not possible.

→ For skew symmetric matrix 3rd order is not possible

System of Linear Equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \quad \text{---} (1)$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_n + \cdots \; a_{nn}x_n = b_n$$

is called system of linear equations in n-variables.

This can be expressed as matrix equations $AX = B$

where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ Coefficient Matrix (A)

$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ Constant matrix.

$\boxed{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ Variable matrix or Unknown matrix.

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \Bigg\} \text{System of Linear Equation in one Variable.}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{bmatrix}$$ Augmented Matrix AB

→ If $B \neq 0$, then $AX = B$ is known as Non-homogenous system of equ.s.
→ If $B = 0$, then $AX = 0$ is known as Homogeneous sys. of equ.s
→ If the system has solution, then it is called consistent system.
→ If the system has no solution, then it is known as inconsistent sys.
→ The zero solution ($x_1, x_2, x_3 = 0$) is called trivial solution.
→ A non zero solution is called non-trivial solution.

Non-Homogeneous System

$AX = B$

① If $\rho(A) \neq \rho(AB)$ then system is inconsistent. No solution
② If $\rho(A) = \rho(AB)$ then system is consistent solution exists.

consistent
$\boxed{\rho(A) = \rho(AB)}$

☆ $\boxed{n = \text{no. of Variables}}$

n is not matrix order but no. of ~~equations~~ variables.

$\boxed{\rho(A) = \rho(AB) = n}$
unique solution.

$\boxed{\text{If } \rho(A) = \rho(AB) < n}$
Infinite no. of Solutions.

If system has unique solution then the no. of independent solution is zero. $n - r = n - n = 0$

No. of independent solution
- $\rho(A) = \rho(AB) = r$
- $(n-r)$ independent solution
- ∞ dependent solutions

Homogeneous System ($AX = 0$)

• It is always consistent system, since it has zero solution (trivial solution).
$\rho(A)$ always $= \rho(AB)$ $B = 0$ Matrix

$\boxed{n - \text{no. of variables}}$

① $\boxed{\rho(A) = n}$ only trivial solution

② $\boxed{\rho(A) < n ; \rho(A) = r}$ Infinite no. of solutions. Including both trivial & Non trivial Solutions.

If A is a square matrix of order $n \times n$

If $|A| \neq 0$

I) if $\rho(A) = n$ ie A is non singular

then it has unique solution which is trivial solution.

II) if $|A| = 0$ then system has infinite no. of solutions.

11.) $x - 2y + z = 3$

$2x + \alpha z = -2$

$-2x + 2y + \alpha z = 1$

$\rho(A) = \rho(AB) = 3$

$\Rightarrow |A| \neq 0$

$$\cancel{A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 0 & \alpha \\ -2 & 2 & \alpha \end{bmatrix}}$$

$\cancel{|A| = \alpha^2 + 4 + 2(2\alpha - 2) + 1(4 + 2\alpha)}$

$\cancel{\Rightarrow |A| = \alpha^2 + 4 + 4\alpha - 4 + 4 + 2\alpha}$

$\cancel{|A| = \alpha^2 + 6\alpha + 4 \neq 0}$

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 0 & \alpha \\ -2 & 2 & \alpha \end{bmatrix}$$

$|A| = 1(-2\alpha) + 2(2\alpha + 2\alpha) + 1(4) \neq 0$

$\Rightarrow -2\alpha + 8\alpha + 4 \neq 0$

$6\alpha + 4 \neq 0$

$\alpha \neq \frac{-4}{6} \Rightarrow \alpha \neq -2/3$

option a

12) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 4 & 3 & 10 \end{bmatrix}$

$|A| = 1(-10 - 12) - 2(20 - 16) + 3(6 + 4)$

$|A| = -22 - 8 + 30 = 0$

Rank $= 2$

A^{-1} doesn't exist

$|A| = 0 \Rightarrow$ Infinite Solution

$\therefore$ option c

13) $AB = \begin{bmatrix} k & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{bmatrix}$ symmetric.

$|A| = k(k^2 - 1) - 1(k - 1) + 1(1 - k)$

$-2k + 2 + k^3 - k \neq 0$

$-3k + 2 + k^3 \neq 0$ $\quad k \neq 1$

option b

17.) $AX = B$

A has linearly indepent column

B is a linear combination of column of A

→ Rank $A = 3$ $|A| = 0$ because of independent column

→ Rank of $A|B$ is 3 [no matter B] because minor of $A|B$ ie A (non Vanishing)

$\therefore$ ~~Ra~~ $\rho(A) = \rho(A|B) = 3 = n$ $\therefore$ Unique Solution. option a

---

18.) $3x + 2y = 1$

$4x + 7z = 1$

$x + y + z = 3$

$x - 2y + 7z = 0$

$A = \begin{bmatrix} 3 & 2 & 0 \\ 4 & 0 & 7 \\ 1 & 1 & 1 \\ 1 & -2 & 7 \end{bmatrix}$ $B = \begin{matrix} 1 \\ 1 \\ 3 \\ 6 \end{matrix}$

$[A|B] = \begin{bmatrix} 3 & 2 & 0 & 1 \\ 4 & 0 & 7 & 1 \\ 1 & 1 & 1 & 3 \\ 1 & -2 & 7 & 0 \end{bmatrix}$ $R_3 \leftrightarrow R_1$ $\begin{bmatrix} 1 & 1 & 1 & 3 \\ 4 & 0 & 7 & 1 \\ 3 & 2 & 0 & 1 \\ 1 & -2 & 7 & 0 \end{bmatrix}$ $R_4 - R_1$ $\begin{bmatrix} 1 & 1 & 1 & 3 \\ 4 & 0 & 7 & 1 \\ 3 & 2 & 0 & 1 \\ 0 & -3 & 6 & -3 \end{bmatrix}$

$R_3 - 3R_1$
$R_2 - 4R_1$ $\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -4 & 3 & -11 \\ 0 & -1 & -3 & -8 \\ 0 & -3 & 6 & -3 \end{bmatrix}$

~~$R_3 \to R_3 + R_1$~~
~~$R_4 \to R_4 + 3R_1$~~ ~~$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -4 & 3 & -11 \\ 0 & 0 & -2 & -5 \\ 0 & 0 & 9 & 6 \end{bmatrix}$~~

~~$R_4 \to 2R_4 + 9R_3$~~ ~~$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -4 & 3 & -11 \\ 0 & 0 & -2 & -5 \\ 0 & 0 & 0 & -33 \end{bmatrix}$~~

~~Rank $A|B = 4$~~
~~Rank $A = 3$~~

$\therefore$ ~~Infinite~~ No solution.

$R_2 \leftrightarrow R_3$ $\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -1 & -3 & -8 \\ 0 & -4 & 3 & -11 \\ 0 & -3 & 6 & -3 \end{bmatrix}$

$R_3 \to R_3 - 4R_1$
$R_4 \to R_4 - 3R_1$ $\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -1 & -3 & -8 \\ 0 & 0 & 15 & 21 \\ 0 & 0 & 15 & 21 \end{bmatrix}$

$R_4 \to R_4 - R_3$ $\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -1 & -3 & -8 \\ 0 & 0 & 15 & 21 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

if no. of eq. less than no. of Variable need not be infinite Solution. They may be inconsistent

Rank $(A|B) = $ Rank $(A) = 3$

$3 = $ no. of Variables $\therefore$ Unique Solution

option C

Eigen Values & Eigen Vectors

Let $A_{n\times n}$ be a square Matrix & $I_{n\times n}$ be unit Matrix & λ be a scalar

⇒ then $\boxed{A-\lambda I \text{ is called Eigen Matrix / Characteristic Matrix / Latent Matrix}}$

⇒ $\boxed{|A-\lambda I| \text{ is called Eigen determinant or Eigen Polynomial.}}$

$\boxed{|A-\lambda I| = 0 \text{ is called eigen equation or characteristic equation.}}$

Eigen equation of 2x2 Matrix Directly

$$\boxed{\lambda^2 - (\text{Trace of } A)\lambda + |A| = 0}$$

Eigen equation for 3x3 Matrix

$$\boxed{\lambda^3 - (\text{Trace of } A)\lambda^2 + (M_{11}+M_{22}+M_{33})\lambda - |A| = 0}$$

Q) Eigen equation of A?

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 2 & -1 \\ 3 & 0 & 2 \end{bmatrix}$$

$$\lambda^3 - 6\lambda^2 + (4 + (-5) + 8)\lambda - [(2\cdot4) + 1(11) + 3(-6)]$$

$$\lambda^3 - 6\lambda^2 + 7\lambda - 1 = 0$$

Eigen Values

The solutions or roots of an eigen equation are known as Eigen roots or characteristic roots.

eg) $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$

$$\lambda^2 - 6\lambda + 5 = 0$$

$$(\lambda-5)(\lambda-1)$$

$$\lambda = 5, 1$$

Eigen Vector

An non zero vector $\boxed{X \neq 0}$ is such that $AX = \lambda X$ or $(A - \lambda I)X = 0$ is called an eigen vector of matrix A corresponding to an eigen value λ

let $B = A - \lambda I$ Eigen Matrix

then $BX = 0$ $X = 0$ always a solution

but for non zero solution $|B| = 0$

$\therefore |A - \lambda I| = 0$

which are satisfied only for Eigen values λ_1, λ_2 etc.

for eigen values there can be corresponding eigenvector.

eg.) $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$ Eigen vector corresponding to 1

$\Rightarrow (A - \lambda I)X = 0$

$(A - I)X = 0$

$\begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = 0 \Rightarrow 3x_1 + 3y_1 = 0$

$x_1 + y_1 = 0$

$\Rightarrow x_1 = -y_1$

$X_1 \therefore = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} k \\ -k \end{bmatrix}$ 1 Independent Solution

∞ dependent Solution.

Eigen Vector corresponding to 5

$(A - 5I)X = 0$

$\begin{bmatrix} -1 & 3 \\ 1 & -3 \end{bmatrix} \Rightarrow x_1 = 3y_1$ $X_5 = \begin{bmatrix} 3k \\ k \end{bmatrix}$ 1 Independent

∞ dependent

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Algebraic Multiplicity (AM)

The number of repetition of an eigen value or order of eigen value is known as Algebraic Multiplicity of eigen value λ.

Geometric Multiplicity (GM).

The no. of linear independent eigen vectors corresponding to λ is geometric multiplicity of λ.
$\boxed{G.M = n - r \text{ in } (A - \lambda I)X = 0}$

Note:
$\boxed{GM \leq AM}$

If $\left.\begin{array}{l} AM = 1 \\ GM = 1 \end{array}\right\}$ distinct eigen values $\Rightarrow$ 1 set of linearly independent / eigen value.

All distinct eigen values $\Rightarrow$ All eigen vectors are independent.

- If AM & GM of ~~every~~ eigen value is same then the matrix is diagonalisable.

Inner product of Vectors

$\boxed{X \cdot Y = X^T Y}$

Inner product of two eigen vectors X & Y is given by $X \cdot Y = X^T Y$ or $Y^T X$

eg: $x = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$ $y = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ then $x \cdot y =$ $1 - 2 + 3 = 2$

Orthogonal Vectors

two vectors X & Y are orthogonal vectors to each other if $X \cdot Y = 0$
$\boxed{\text{ie inner product is zero}}$
then $X \perp Y$

eg. $X = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ $Y = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ $X \cdot Y = -1 + 0 + 1 = 0$

X & Y are orthogonal vectors.
$x \perp Y$

> If a real symmetric matrix having distinct eigen values, then its eigen vectors are pair wise orthogonal vectors.

Properties of Eigen Values

1) Sum of Eigen Values = trace of A
2) Product of Eigen values = $\det(A) = |A|$
3) If A is singular, then any one (atleast) (eigenvalue) must be zero & vice versa
4) Eigen values of diagonal, scalar, & triangular matrices are the diagonal elements itself.

Eigen values of real symmetric matrix are real numbers

5.) Eigen values of skew symmetric matrix are either zeroes or purely imaginary. ~~(Nsquare of terms)~~
6) Eigen values of a orthogonal matrix are of unit modulus
7) Eigen values of a orthogonal matrix lies on unit circle.

ie $|\lambda| = 1$ eigen value lies on unit circle.

8) Eigen values of nilpotent matrix are zeroes (trace zero for nilpotent $|A| = 0$)
9.) Eigen Values of A & A^T are same.

If λ is eigen value of A

then

① $k\lambda$ is eigen value of kA
② λ^m is eigen value of A^m
③ $\lambda \pm k$ is eigen value of $A \pm kI$
④ $1/\lambda$ is eigen value of A^{-1}
⑤ $\frac{|A|}{\lambda}$ is eigen value of Adjoint A

Properties of Eigen Vectors

1) $X \neq 0$
2) $\boxed{\text{Eigen vectors of } A \text{ \& } A^T \text{ are not same}}$
3) Eigen Vectors of A, kA, A^m, A^{-I} are same
4) One eigen value corresponds to more than one eigen vectors but one eigen vector cannot corresponds to more than one eigen value.

Cayley Hamilton theorm

→ Every square matrix satisfies it's own characteristic equation.

ie) if λ is replaced by A in eigen equation. then it is true & satisfies.

eg:- $\lambda^3 - 3\lambda^2 + 5\lambda - 7 = 0$

$\Rightarrow A^3 - 3A^2 + 5A - 7I = 0$

Minimal Polynomial

A polynomial consisting of minimum degree which satisfies the matrix A is called minimal polynomial of A.

→ If a matrix has distinct eigen values, then characteristic polynomial & minimal polynomial are same.

→ The degree of minimal polynomial greater than or equal to the number of distinct eigen values. (factor corresponding to distinct eigen value is mandatory $(\lambda-\lambda_0)(\lambda-\lambda_1)(\lambda-\lambda_2)$ $\lambda_0, \lambda_1, \lambda_2$ distinct.)

→ A minimal polynomial should satisfy all eigen values of A including A.

---

19) ϕ $A' \to 0, 1, 3$ dosen't exist } so can't say.
$A^{-1} \to 1, 2, 3$ exist } same for Rank A = 3

∴ option d.

---

20) $|A| = 8(21-16) + 6(-18+8) + 2(24-14)$

$|A| \quad 8(5) + 6(-10) + 2 \times 10$

$|A| = 40 - 60 + 20 = 0$

trace = sum of eign = 18

product ~~of~~ eigen value = 0

∴ option b

$\lambda^3 - 18\lambda^2 + (5+20+20)\lambda - 0 = 0$

$\lambda^3 - 18\lambda^2 + 45\lambda = 0$

$\lambda(\lambda^2 - 18\lambda + 45) = 0$

$\lambda(\lambda-15)(\lambda-3) = 0$

∴ 0, 15, 3 //

21.) ★

$\begin{bmatrix} a & a \\ a & a \end{bmatrix}_{3\times3}$

$\lambda^2 - 2a\lambda = 0$

$\lambda(\lambda - 2a)$

0, 2a

$\begin{bmatrix} a & a & a \\ a & a & a \\ a & a & a \end{bmatrix}_{3\times3}$

$\lambda^3 - 3a\lambda^2 = 0$

$\lambda^2(\lambda - 3a)$

0, 0, 3a

$\begin{bmatrix} a & \cdots & a \\ \vdots & & \\ a & \cdots & a \end{bmatrix}_{n\times n}$

$\lambda^n - na\lambda^{n-1}$

$\lambda^{n-1}(\lambda - na)$

000 ... n time, na

$\therefore A = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix}_{n\times n} = \underbrace{0 \cdots}_{n\text{ time}}, n$ distinct $= 0, n$.

22. $C.E = (\lambda-1)(\lambda-2)(\lambda-3)$

$\Rightarrow (A-I)(A-2I)(A-3I) = 0$

$(A^2 - 3A + 2I)(A - 3I)$

$= A^3 - 3A^2 - 3A^2 + 9A + 2A - 6I = 0$

$\Rightarrow A^3 - 6A^2 + 11A - 6I = 0$

directly $\boxed{\lambda^3 - (\lambda_1 + \lambda_2 + \lambda_3)\lambda^2 + (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)\lambda - \lambda_1\lambda_2\lambda_3 = 0}$

$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$A^3 - 6A^2 + 11A - 6I = 0$

$A^2 - 6A + 11I - 6A^{-1} = 0$

$6A^{-1} = \underline{A^2 - 6A + 11I}$

23) $\begin{bmatrix} 10 & -4 \\ 18 & -12 \end{bmatrix}$

$\lambda^2 + 2\lambda + (-120 + 72) = \lambda^2 + 2\lambda - 48 = 0$

$(\lambda + 8)(\lambda - 6) = 0$

$\lambda = \underline{-8, 6}$

$(A - \lambda I)X = 0$

$\begin{pmatrix} 18 & -4 \\ 18 & -4 \end{pmatrix} X = 0$

$18x = 4y$

$9x = 2y$

$X = \begin{bmatrix} 2k \\ 9k \end{bmatrix}$ $\therefore$ option b.

$$\begin{bmatrix}1&0&0&3\\0&2&1&3\\0&1&-1&-3\\0&3&0&0\end{bmatrix} \rightarrow \begin{bmatrix}1&0&0&3\\0&1&-1&-3\\0&2&1&3\\0&3&0&0\end{bmatrix} \rightarrow \begin{bmatrix}1&0&0&3\\0&1&-1&-3\\0&0&3&9\\0&0&3&9\end{bmatrix} \rightarrow \begin{bmatrix}1&0&0&3\\0&1&-1&-3\\0&0&3&9\\0&0&0&0\end{bmatrix}$$

$\therefore$ Rank = 3.

a) $|A| = 0$ ✓

b) Rank of $A = 3$ ✓

c) $AX = 0$ has infinite solution $\therefore$ non zero ✓

d) $AX = B$ has infinite solution ✗ optional

Some Important Matrix Results.

Trace Properties

- $tr(A \pm B) = tr(A) \pm tr(B)$
- $tr(AB) \neq tr(A) \cdot tr(B)$
- $tr(AB) = tr(BA)$

Diagonal Matrix ~~if A, B diagonal Matrix~~ then

- $A \pm B$ diagonal
- A^2, B^2 diagonal
- A^{-1}, B^{-1} diagonal
- A^T diagonal
- Adj(A), Adj B diagonal

Transpose properties

- $\boxed{(A \pm B)^T = A^T \pm B^T}$
- $(AB)^T = B^T A^T$
- $\boxed{(ABCD)^T = D^T C^T B^T A^T}$
- $\Rightarrow \boxed{(A^n)^T = (A^T)^n}$

Symmetric & Skew Symmetric Not Written properties

if A, B symmetric then

- $A \pm B$ symmetric
- AB & $BA \neq$ sym.
- $\boxed{\begin{matrix}AB + BA & \text{symm.}\\ AB - BA & \text{skew sym}\end{matrix}}$
- A^k, B^k symmetric $(k \in N)$

if A & A^T are square then

- $A + A^T =$ symm.
- $A - A^T, A^T - A$ – skew sy
- $\boxed{AA^T, A^TA \text{ – symm}}$

if A & B are skew symmetric then

- $A \pm B =$ skew sy.
- A^2, B^2 are symm. ie A^{even}
- A^3, B^3 are skew sy. ie A^{odd}

if A & B are symmetric or skew symmetric then $AB = BA$

* if A & B are either symmetric or skew symmetric then AB & BA are also orthogonal.

* if A & B are orthogonal then

* if A & B are

- Multiplication Associative

eg. $ABC = A(BC) = (AB)C$

- Multiplication distributive over addition
- eg. $A(B+C) = AB + AC$
- eg. $(B+C)A = BC + CA$

- if A & B are non zero then $AB = 0 \Rightarrow A$ & B singular
- if A non zero then $AB = 0 \Rightarrow B = 0$

★ The each element of a row (or column) of a determinant if expressed as a sum of 2 or more terms then the determinant can be expressed as the sum of two or more determinants

eg:-
$$\begin{vmatrix}a+d_1 & b+d_2 & c+d_3\\ d & e & f\\ g & h & i\end{vmatrix} = \begin{vmatrix}a & b & c\\ d & e & f\\ g & h & i\end{vmatrix} + \begin{vmatrix}d_1 & d_2 & d_3\\ d & e & f\\ g & h & i\end{vmatrix}$$

Properties of Adjoint.

- $\boxed{A \cdot Adj(A) = adj(A) \cdot A = |A|}$ ★
- If A is diagonal then $adj(A)$ also diagonal.

$$A = \begin{bmatrix} l & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & n \end{bmatrix} \quad AdjA = \begin{bmatrix} mn & 0 & 0 \\ 0 & ln & 0 \\ 0 & 0 & lm \end{bmatrix}$$

- $\boxed{Adj(A^T) = Adj(A)^T}$ ★

$Adj(A^T) = |A^T| \cdot (A^T)^{-1}$ | Since $|A^T| = |A|$ &
$Adj(A^T) = |A| \cdot (A^{-1})^T$ | $(A^{-1})^T = (A^T)^{-1}$
$Adj(A^T) = Adj(A)^T$

- $\boxed{adj(AB) = adj(A) \cdot adj B}$ ★
- $A = \text{symmetric} \Rightarrow Adj A = \text{Symmetric}$
- $\boxed{Adj(k A_{n\times n}) = k^{n-1} Adj(A_{n\times n})}$

$Adj(k A_{n\times n}) = |k A_{n\times n}| \cdot (k A_{n\times n})^{-1}$

$Adj\, k(A_{n\times n}) = k^n |A_{n\times n}| \cdot \frac{1}{k}(A_{n\times n})^{-1}$

Since $|kA| = k^n|A|$ & $(kA)^{-1} = \frac{1}{k}A^{-1}$

$\therefore Adj(k A_{n\times n}) = k^{n-1} Adj(A_{n\times n})$

Properties of Inverse

- $\boxed{(A^T)^{-1} = (A^{-1})^T}$
- $\boxed{(ABCD)^{-1} = D^{-1}C^{-1}B^{-1}A^{-1}}$
- $A \to$ symmetric $\boxed{A^{-1} \text{ symmetric}}$ ★
- If $AB = BA$ then $A^{-1}B^{-1} = B^{-1}A^{-1}$
- If A orthogonal then A^T & A^{-1} orthogonal
- $\boxed{kA^{-1} = \frac{1}{k}A^{-1}}$

$X \cdot Y = X^T Y = Y^T X$

$X \cdot Y = 0 \Rightarrow$ orthogonal / perpendicular vector

$X \cdot Y = \pm 1 \Rightarrow$ parallel vector.

$\sqrt{XX^T} = \sqrt{X \cdot X} = \begin{pmatrix} \text{length of vector } X \\ \text{norm of } \quad '' \quad X \end{pmatrix}$

$\|X\| = \sqrt{X \cdot X} = \sqrt{XX^T} = 1 \Rightarrow$ normal vector

- If orthogonal each other & normal to self then orthonormal

ie $X_1, X_2, X_3 \ldots$ of same order

then $\boxed{X_i^T X_j = \begin{cases} 0, & \forall i \neq j \\ 1, & \forall i = j \end{cases}}$

- If λ is eigen value of orthogonal matrix $\Rightarrow$ $1/\lambda$ also eigen value
- $a_0 + a_1\lambda + a_2\lambda^2 + \ldots$ is an eigen value of $B = a_0 I + a_1 A + a_2 A^2 + \ldots$
- If matrix satisfies an equation then eigen values also satisfy
- $\frac{|A|}{\lambda}$ is eigen value of $(Adj A)$
- If $a + \sqrt{b}$ is eigen value then $a - \sqrt{b}$ also eigen value
- If $a + jb$ is eigen value then $(a - jb)$ also eigen value.
- Eigen Vectors of A & $a_0 I + a_1 A + a_2 A^2 \ldots\ldots$ are same.

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- If $X_1 X_2 X_3$ are linearly independent vectors for $\lambda_1 \lambda_2 \lambda_3$ then $P = [X_1\ X_2\ X_3] \Rightarrow \boxed{P^{-1}AP = D}$ $\boxed{P^{-1}A^kP = D^k}$

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

or $\boxed{\begin{array}{l} A = PDP^{-1} \\ A^k = PD^kP^{-1} \end{array}}$

- The polynomial of lowest degree that annihilates A is called minimal poly.
- Minimal poly perfectly divides C.E polynomial
- Minimal poly satisfies both C.E & Minimal poly.
- λ eigen value satisfies both C.E & Minimal poly.
  - degree of Min poly $= d$
  - No. of disting eigen value $= \ell$
  - degree of C.E $= \epsilon$

$$\boxed{\ell \le d \le \epsilon}$$

- For matrix equation Post multiplication & pre multiplication must be done likewise on both sides while solving.
- To check whether vectors are linearly independent $\Rightarrow$ form matrix find rank.
- To find rank, need not always convert to row echelon. Rank property & or definition can be applied (eg if $|A| = 0$, then rank $< n$)
- When A given and equation of A with higher power asked find C.E.
  eg $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ find A^9 (Sol $A^9 = 625A$)
- If a matrix is defined in terms of n, i, j by some equation & some property is asked it is always feasible to create a representative of the matrix by assuming values.
- zero eigen value $\Rightarrow |A| = 0$ & vice versa
- distinct eigen value $\Rightarrow$ eigen vectors linearly independent
-

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